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Finite Energy Electroweak Monopoles

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Abstract

We present finite energy analytic monopole and dyon solutions whose size is fixed by the electroweak scale. The new solutions are obtained by regularizing the recent Cho-Maison solutions of the Weinberg-Salam model. Our result shows that genuine electroweak monopole and dyon could exist whose mass scale is much smaller than the grand unification scale.

I. Introduction

Ever since Dirac [1] has introduced the concept of the magnetic monopole, the monopoles have remained a fascinating subject in theoretical physics. The Abelian monopole has been generalized to the non-Abelian gauge theory by Wu and Yang [2] who showed that the pure $SU(2)$ gauge theory allows a point-like monopole, and by 't Hooft and Polyakov [3] who have constructed a finite energy monopole solution in the Georgi-Glashow model as a

topological soliton. In the interesting case of the electroweak theory of Weinberg and Salam, however, it has generally been believed that there exists no topological monopole of physical interest. The basis for this “non-existence theorem” is, of course, that with the spontaneous symmetry breaking the quotient space $SU(2) \times U(1)/U(1)_{\text{em}}$ allows no non-trivial second homotopy. This has led many people to conclude that there is no topological structure in the Weinberg-Salam model which can accommodate a magnetic monopole.

This, however, is shown to be not true. Indeed recently Cho and Maison [4] have established that the Weinberg-Salam model has exactly the same topological structure as the Georgi-Glashow model which allows the magnetic monopoles, and demonstrated the existence of a new type of monopole and dyon solutions in the standard Weinberg-Salam model. *This was based on the observation that the Weinberg-Salam model, with the hypercharge $U(1)$, could be viewed as a gauged CP^1 model in which the (normalized) Higgs doublet plays the role of the CP^1 field.* So the Weinberg-Salam model does have exactly the same nontrivial second homotopy as the Georgi-Glashow model which allows topological monopoles. Once this is understood one could proceed to construct the desired monopole and dyon solutions in the Weinberg-Salam model. Originally the Cho-Maison solutions were obtained by a numerical integration, but now a mathematically rigorous existence proof has been established which supports the numerical results [5].

The Cho-Maison monopole may be viewed as a hybrid between the Dirac monopole and the 't Hooft-Polyakov monopole, because it has a $U(1)$ point singularity at the center even though the $SU(2)$ part is completely regular. Consequently it carries an infinite energy so that at the classical level the mass of the monopole remains arbitrary. *A priori* there is nothing wrong with this, but nevertheless one may wonder whether one can have an analytic electroweak monopole which has a finite energy. The purpose of this paper is to show that this is indeed possible, and to present explicit electroweak monopole solutions with finite energy. Clearly the new monopoles should have important physical applications in the phenomenology of electroweak interaction.

II. Monopoles in Weinberg-Salam Model

Before we construct the finite energy monopole solutions we must understand how one could obtain the infinite energy solutions first. So we will briefly review the Cho-Maison solutions in the Weinberg-Salam model. Let us start with the Lagrangian which describes (the bosonic sector of) the standard Weinberg-Salam model

$$\mathcal{L} = -|\hat{D}_\mu\phi|^2 - \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{\mu^2}{\lambda}\right)^2 - \frac{1}{4}(\mathbf{F}_{\mu\nu})^2 - \frac{1}{4}(G_{\mu\nu})^2, \quad (2.1)$$

$$\hat{D}_\mu\phi = \left(\partial_\mu - i\frac{g}{2}\boldsymbol{\tau}\cdot\mathbf{A}_\mu - i\frac{g'}{2}B_\mu\right)\phi = \left(D_\mu - i\frac{g'}{2}B_\mu\right)\phi,$$

where ϕ is the Higgs doublet, $\mathbf{F}_{\mu\nu}$ and $G_{\mu\nu}$ are the gauge field strengths of $SU(2)$ and $U(1)$ with the potentials \mathbf{A}_μ and B_μ , and g and g' are the corresponding coupling constants. Notice that D_μ describes the covariant derivative of the $SU(2)$ subgroup only. From (2.1) one has the following equations of motion

$$\begin{aligned} \hat{D}_\mu(\hat{D}_\mu\phi) &= \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{\lambda}\right)\phi, \\ D_\mu\mathbf{F}_{\mu\nu} &= -\mathbf{j}_\nu = i\frac{g}{2}\left[\phi^\dagger\boldsymbol{\tau}(\hat{D}_\nu\phi) - (\hat{D}_\nu\phi)^\dagger\boldsymbol{\tau}\phi\right], \\ \partial_\mu G_{\mu\nu} &= -k_\nu = i\frac{g'}{2}\left[\phi^\dagger(\hat{D}_\nu\phi) - (\hat{D}_\nu\phi)^\dagger\phi\right]. \end{aligned} \quad (2.2)$$

Now we choose the following static spherically symmetric ansatz

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2}}\rho(r)\xi(\theta, \varphi), \\ \xi &= i\begin{pmatrix} \sin(\theta/2) e^{-i\varphi} \\ -\cos(\theta/2) \end{pmatrix}, \quad \hat{\phi} = \xi^\dagger\boldsymbol{\tau}\xi = -\hat{r}, \\ \mathbf{A}_\mu &= \frac{1}{g}A(r)\hat{\phi}\partial_\mu t + \frac{1}{g}(f(r) - 1)\hat{\phi} \times \partial_\mu\hat{\phi}, \end{aligned} \quad (2.3)$$

$$B_\mu = -\frac{1}{g'}B(r)\partial_\mu t - \frac{1}{g'}(1 - \cos\theta)\partial_\mu\varphi,$$

where (t, r, θ, φ) are the spherical coordinates. Notice that the apparent string singularity along the negative z-axis in ξ and B_μ is a pure gauge artifact which can easily be removed with a hypercharge $U(1)$ gauge transformation. Indeed one can easily excise the string by making the hypercharge $U(1)$ bundle non-trivial [2]. So *the above ansatz describes a most general spherically symmetric ansatz of a $SU(2) \times U(1)$ dyon.* Here we emphasize the importance of the non-trivial $U(1)$ degrees of freedom to make the ansatz spherically symmetric. Without the extra $U(1)$ the Higgs doublet does not allow a spherically symmetric ansatz. This is because the spherical symmetry for the gauge field involves the embedding of the radial isotropy group $SO(2)$ into the gauge group that requires the Higgs field to be invariant under the $U(1)$ subgroup of $SU(2)$. This is possible with a Higgs triplet, but not with a Higgs doublet [6]. In fact, in the absence of the hypercharge $U(1)$ degrees of freedom, the above ansatz describes the $SU(2)$ sphaleron which is not spherically symmetric [7]. The situation changes with the inclusion of the extra hypercharge $U(1)$ in the standard model, which can compensate the action of the $U(1)$ subgroup of $SU(2)$ on the Higgs field.

The spherically symmetric ansatz (2.3) reduces the equations of motion to

$$\begin{aligned} \ddot{f} - \frac{f^2 - 1}{r^2}f &= \left(\frac{g^2}{4}\rho^2 - A^2\right)f, \\ \ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{f^2}{2r^2}\rho &= -\frac{1}{4}(B - A)^2\rho + \lambda\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)\rho, \\ \ddot{A} + \frac{2}{r}\dot{A} - \frac{2f^2}{r^2}A &= \frac{g^2}{4}\rho^2(A - B), \\ \ddot{B} + \frac{2}{r}\dot{B} &= \frac{g'^2}{4}\rho^2(B - A). \end{aligned} \tag{2.4}$$

The smoothness of the solution requires the following boundary conditions near the origin,

$$\begin{aligned} f &\simeq 1 + \alpha_1 r^2, \\ \rho &\simeq \beta_1 r^\delta, \\ A &\simeq a_1 r, \\ B &\simeq b_0 + b_1 r, \end{aligned} \tag{2.5}$$

where $\delta = (-1 + \sqrt{3})/2$. On the other hand asymptotically the finiteness of energy requires the following condition,

$$\begin{aligned} f &\simeq f_1 \exp(-\kappa r), \\ \rho &\simeq \rho_0 + \rho_1 \frac{\exp(-\sqrt{2}\mu r)}{r}, \\ A &\simeq A_0 + \frac{A_1}{r}, \\ B &\simeq A + B_1 \frac{\exp(-\nu r)}{r}, \end{aligned} \tag{2.6}$$

where $\rho_0 = \sqrt{2\mu^2/\lambda}$, $\kappa = \sqrt{(g\rho_0)^2/4 - A_0^2}$, and $\nu = \sqrt{(g^2 + g'^2)\rho_0}/2$. Notice that asymptotically $B(r)$ must approach to $A(r)$ with an exponential damping.

To determine the electric and magnetic charge of the dyon we now perform the following gauge transformation on (2.3)

$$\xi \longrightarrow \xi' = U\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.7}$$

$$U = i \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\varphi} \\ -\sin(\theta/2)e^{i\varphi} & \cos(\theta/2) \end{pmatrix},$$

and find that in this unitary gauge

$$\mathbf{A}_\mu \longrightarrow \mathbf{A}'_\mu = \frac{1}{g} \begin{pmatrix} -f(r)(\sin\varphi\partial_\mu\theta + \sin\theta\cos\varphi\partial_\mu\varphi) \\ f(r)(\cos\varphi\partial_\mu\theta - \sin\theta\sin\varphi\partial_\mu\varphi) \\ -A(r)\partial_\mu t - (1 - \cos\theta)\partial_\mu\varphi \end{pmatrix}. \tag{2.8}$$

So expressing the electromagnetic potential \mathcal{A}_μ and the neutral potential Z_μ with the Weinberg angle θ_w

$$\begin{aligned} \begin{pmatrix} \mathcal{A}_\mu \\ Z_\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} B_\mu \\ A_\mu^3 \end{pmatrix} \\ &= \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g & g' \\ -g' & g \end{pmatrix} \begin{pmatrix} B_\mu \\ A_\mu^3 \end{pmatrix}, \end{aligned} \tag{2.9}$$

we have

$$\begin{aligned}\mathcal{A}_\mu &= -e\left(\frac{1}{g^2}A + \frac{1}{g'^2}B\right)\partial_\mu t - \frac{1}{e}(1 - \cos\theta)\partial_\mu\varphi, \\ Z_\mu &= \frac{e}{gg'}(B - A)\partial_\mu t,\end{aligned}\tag{2.10}$$

where e is the electric charge

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g \sin\theta_w.$$

From this one has the following electromagnetic charges of the dyon

$$\begin{aligned}q_e &= 4\pi e \left[r^2 \left(\frac{1}{g^2} \dot{A} + \frac{1}{g'^2} \dot{B} \right) \right] \Big|_{r=\infty} = \frac{4\pi}{e} A_1 \\ &= \frac{8\pi}{e} \sin^2\theta_w \int_0^\infty f^2 A dr, \\ q_m &= \frac{4\pi}{e}.\end{aligned}\tag{2.11}$$

Also, from the asymptotic condition (2.6) we conclude that our dyon does not carry any neutral charge,

$$\begin{aligned}Z_e &= -\frac{4\pi e}{gg'} \left[r^2 (\dot{B} - \dot{A}) \right] \Big|_{r=\infty} = 0, \\ Z_m &= 0,\end{aligned}\tag{2.12}$$

which is what one should have expected.

With the boundary conditions one can integrate (2.4) and find the Cho-Maison dyon solution shown in Fig.1 [4]. The regular part of the solution looks very much like the well-known Prasad-Sommerfield solution of the Julia-Zee dyon [8]. But there is a crucial difference. The Cho-Maison dyon now has a non-trivial $B - A$, which represents the non-vanishing neutral Z boson content of the dyon as shown by (2.10). To understand the behavior of the solutions, remember that the mass of the W and Z bosons are given by $M_W = g\rho_0/2$ and $M_Z = \sqrt{g^2 + g'^2}\rho_0/2$, and the mass of Higgs boson is given by $M_H = \sqrt{2}\mu$. This confirms that $\sqrt{(M_W)^2 - (A_0)^2}$ and M_H determines the exponential damping of f and ρ , and M_Z determines the exponential damping of $B - A$, to their vacuum expectation values asymptotically. These are exactly what one would have expected.

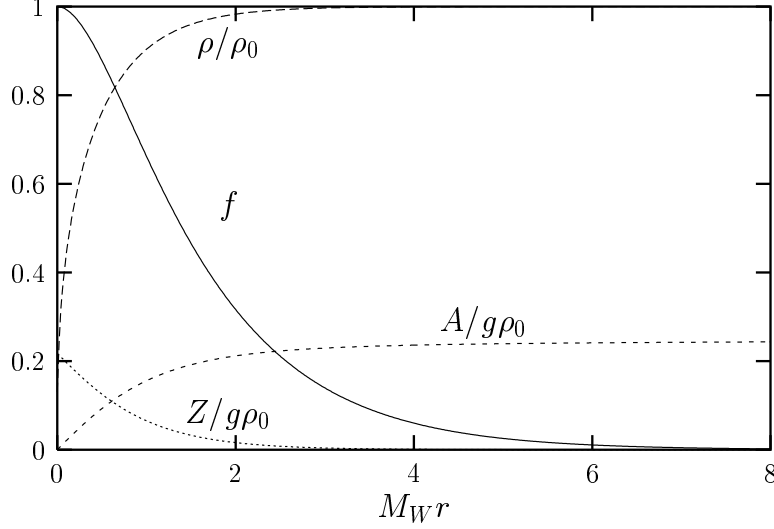


Figure 1: The Cho-Maison dyon solution. Here $Z(r) = B(r) - A(r)$ and we have chosen $\sin^2 \theta_w = 0.2325$, $\lambda/g^2 = M_H^2/2M_W^2 = 0.5$, and $A_0 = M_W/2$.

With the ansatz (2.3) the energy of the dyon is given by

$$E = E_0 + E_1, \quad (2.13)$$

$$E_0 = \frac{2\pi}{g^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{g^2}{r^2} + (1 - f^2)^2 \right\},$$

$$E_1 = \frac{4\pi}{g^2} \int_0^\infty dr \left\{ \frac{g^2}{2} (r\dot{\rho})^2 + \frac{g^2}{4} f^2 \rho^2 + \frac{g^2 r^2}{8} (B - A)^2 \rho^2 + \frac{\lambda g^2 r^2}{2} \left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda} \right)^2 \right. \\ \left. + (\dot{f})^2 + \frac{1}{2} (r\dot{A})^2 + \frac{g^2}{2g'^2} (r\dot{B})^2 + f^2 A^2 \right\}.$$

Now with the boundary conditions (2.5) and (2.6) one could easily find that E_1 is finite. As for E_0 we can minimize it with the boundary condition $f(0) = 1$, but even with this E_0 becomes infinite. Of course the origin of this infinite energy is obvious, which is precisely due to the magnetic singularity at the origin. This means that one can not predict the mass of dyon. It remains arbitrary at the classical level.

III. Analytic Solutions

At this stage one may ask whether there is any way to make the energy of the Cho-Maison solutions finite. A simple way to make the energy finite is to introduce the gravitational interaction [9]. But the gravitational interaction is not likely remove the singularity at the origin, and one may still wonder if there is any way to regularize the Cho-Maison solutions. In this section we will show that this is indeed possible, and discuss how one can construct the monopole and dyon solutions explicitly which have not only a finite energy but also analytic everywhere.

To do this we first notice that a non-Abelian gauge theory in general is nothing but a special type of an Abelian gauge theory which has a well-defined set of charged vector fields as its source. This must be obvious, but this trivial observation reminds us the fact that *the finite energy non-Abelian monopoles are really nothing but the Abelian monopoles whose singularity is regularized by the charged vector fields*. From this perspective one can try to make the energy of the above solutions finite by introducing additional interactions and/or charged vector fields. In the followings we will present two ways which allow us to achieve this goal along this line, and construct analytic electroweak monopole and dyon solutions with finite energy.

A. Electromagnetic Regularization

Remember that the origin of the infinite energy of the Cho-Maison solutions is the magnetic singularity of $U(1)_{\text{em}}$ at the origin. We could try to regularize this singularity with a judicious choice of an extra electromagnetic interaction of the charged vector field with the Abelian monopole. This regularization would provide a most economic way to make the energy of the Cho-Maison solution finite, because here we could use the already existing W boson without introducing a new source.

To show that this is indeed possible we first notice that in the unitary gauge the La-

grangian (2.1) can be written as

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2}|D_\mu W_\nu - D_\nu W_\mu|^2 \\
& + igF_{\mu\nu}W_\mu^*W_\nu + \frac{1}{4}g^2(W_\mu^*W_\nu - W_\nu^*W_\mu)^2 \\
& - \frac{1}{2}(\partial_\mu \rho)^2 - \frac{1}{4}\rho^2(g^2W_\mu^*W_\mu + \frac{1}{2}(g'B_\mu - gA_\mu)^2) - \frac{\lambda}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
W_\mu &= \frac{1}{\sqrt{2}}(A_\mu^1 + iA_\mu^2), \\
A_\mu &= A_\mu^3, \\
D_\mu W_\nu &= (\partial_\mu + igA_\mu)W_\nu.
\end{aligned}$$

This Lagrangian describes the dynamics of two $U(1)$ gauge fields A_μ and B_μ interacting with a charged vector field W_μ and a real scalar field ρ . Notice that in the unitary gauge the spherically symmetric ansatz (2.3) is written as

$$\begin{aligned}
\rho &= \rho(r) \\
W_\mu &= \frac{i}{g} \frac{f(r)}{\sqrt{2}} e^{i\varphi} (\partial_\mu \theta + i \sin \theta \partial_\mu \varphi), \\
A_\mu &= -\frac{1}{g} A(r) \partial_\mu t - \frac{1}{g} (1 - \cos \theta) \partial_\mu \varphi, \\
B_\mu &= -\frac{1}{g'} B(r) \partial_\mu t - \frac{1}{g'} (1 - \cos \theta) \partial_\mu \varphi,
\end{aligned} \tag{3.2}$$

which must be clear from (2.8).

To regularize the Cho-Maison dyon, we now introduce an extra interaction \mathcal{L}_1 to (3.1),

$$\mathcal{L}_1 = i\alpha g F_{\mu\nu} W_\mu^* W_\nu + \frac{\beta}{4} g^2 (W_\mu^* W_\nu - W_\nu^* W_\mu)^2, \tag{3.3}$$

where α and β are arbitrary constants. With this additional interaction the energy of system is given by

$$E = E'_0 + E_1, \tag{3.4}$$

where now E'_0 is given by

$$E'_0 = \frac{2\pi}{g^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{g^2}{g'^2} + 1 - 2(1+\alpha)f^2 + (1+\beta)f^4 \right\}. \quad (3.5)$$

Notice that with $\alpha = \beta = 0$, E'_0 reduces to E_0 and becomes infinite. Clearly, for the energy (3.4) to be finite, E'_0 must be free not only from the $O(1/r^2)$ singularity but also $O(1/r)$ singularity at the origin. This requires us to have

$$\begin{aligned} 1 + \frac{g^2}{g'^2} - 2(1+\alpha)f^2(0) + (1+\beta)f^4(0) &= 0, \\ (1+\alpha)f(0) - (1+\beta)f^3(0) &= 0. \end{aligned} \quad (3.6)$$

Thus we arrive at the following condition in order to have a finite energy

$$\frac{(1+\alpha)^2}{1+\beta} = 1 + \frac{g^2}{g'^2} = \frac{1}{\sin^2 \theta_w}, \quad (3.7)$$

from which we have

$$f(0) = \frac{1}{\sqrt{(1+\alpha)\sin^2 \theta_w}}. \quad (3.8)$$

In general $f(0)$ can assume an arbitrary value depending on the choice of α . But notice that, except for $f(0) = 1$, the $SU(2)$ gauge field is not well-defined at the origin. This means that only when $f(0) = 1$, or equivalently only when $\alpha = \beta$, the solution becomes analytic everywhere including the origin.

Now, the equations of motion that extremise the energy functional are given by

$$\begin{aligned} \ddot{f} - \frac{(1+\beta)f^2 - (1+\alpha)}{r^2} f &= \left(\frac{g^2}{4} \rho^2 - A^2 \right) f, \\ \ddot{\rho} + \frac{2}{r} \dot{\rho} - \frac{f^2}{2r^2} \rho &= -\frac{1}{4} (B-A)^2 \rho + \lambda \left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda} \right) \rho, \\ \ddot{A} + \frac{2}{r} \dot{A} - \frac{2f^2}{r^2} A &= \frac{g^2}{4} (A-B) \rho^2, \\ \ddot{B} + \frac{2}{r} \dot{B} &= \frac{g'^2}{4} (B-A) \rho^2. \end{aligned} \quad (3.9)$$

One could integrate this with the boundary conditions

$$\begin{aligned} f(0) &= 1/\sqrt{(1+\alpha)\sin^2\theta_w}, \quad A(0) = 0, \quad B(0) = b_0, \quad \rho(0) = 0, \\ f(\infty) &= 0, \quad A(\infty) = B(\infty) = A_0, \quad \rho(\infty) = \rho_0. \end{aligned} \quad (3.10)$$

Notice that since E_1 contains the term $r^2(B-A)^2\rho^2$, one must have $A(\infty) = B(\infty)$ to make the energy finite. Moreover notice that (3.9) is invariant under $(A, B) \rightarrow (-A, -B)$. From this symmetry and the last two equations of (3.9) one can show that $B(r) \geq A(r) \geq 0$ for all range [5]. This tells us that b_0 can not be negative. The results of the numerical integration for the monopole and dyon solution are shown in Fig.2 and Fig.3. Here we have chosen the experimental value of $\sin^2\theta_w (= 0.2325)$, and assumed $f(0) = 1$ to guarantee the analyticity of the solution. *It is really remarkable that the finite energy solutions look almost identical to the Cho-Maison solutions, even though they no longer have the singularity at the origin and analytic everywhere.*

Clearly the energy of the above solutions must be of the order of M_W . Indeed for the monopole the energy can be expressed as

$$E = \frac{4\pi}{e^2} C(\alpha, \sin^2\theta_w, \lambda/g^2) M_W \quad (3.11)$$

where C the dimensionless function of α , $\sin^2\theta_w$, and λ/g^2 . With $f(0) = 1$ and experimental value $\sin^2\theta_w$, C becomes slowly varying function of λ/g^2 with $C = 0.540$ for $\lambda/g^2 = 0$. This demonstrates that the finite energy solutions can indeed be interpreted as the electroweak monopole and dyon, and are really nothing but the regularized Cho-Maison solutions which have a mass of the electroweak scale.

It is interesting to notice that for the monopole solution we can find the Bogomol'nyi-type energy bound if we add an extra term \mathcal{L}_2 in the Lagrangian (3.1) and (3.3)

$$\mathcal{L}_2 = -\left((1+\alpha)^2\sin^2\theta_w - \frac{1}{4}\right)g^2\rho^2W_\mu^*W_\mu. \quad (3.12)$$

Notice that this amounts to changing the mass of the W boson from $g\rho_0/2$ to $(1+\alpha)e\rho_0$. In this case with (3.7) the energy functional in the Prasad-Sommerfield limit $\lambda = 0$ becomes

$$E = \int d^3x \left\{ \frac{1}{2} |D_i W_j - D_j W_i|^2 + \frac{1}{2} \left[i(1+\alpha)e\epsilon_{ijk} W_j^* W_k - \frac{g}{2e} \epsilon_{ijk} F_{jk} \right]^2 \right\}$$

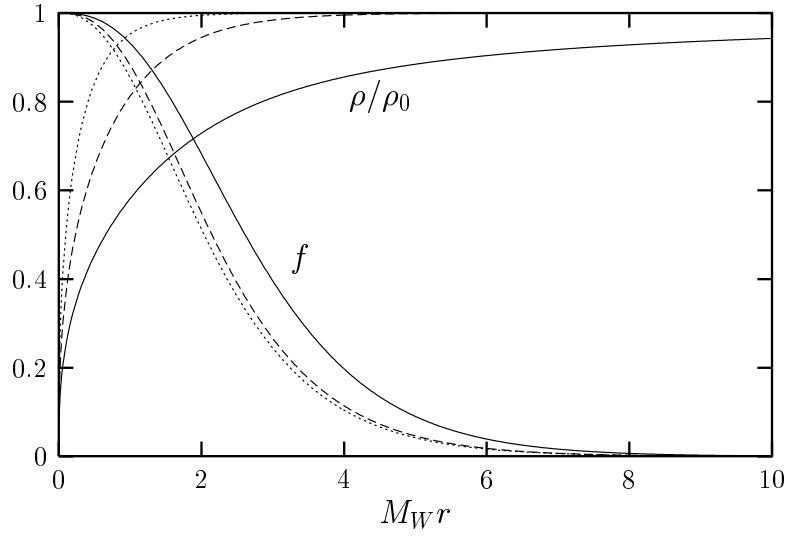


Figure 2: The finite energy electroweak monopole solution obtained with different values of $\lambda/g^2 = 0$ (solid line), 0.5(dashed line), and 4.5(dotted line).

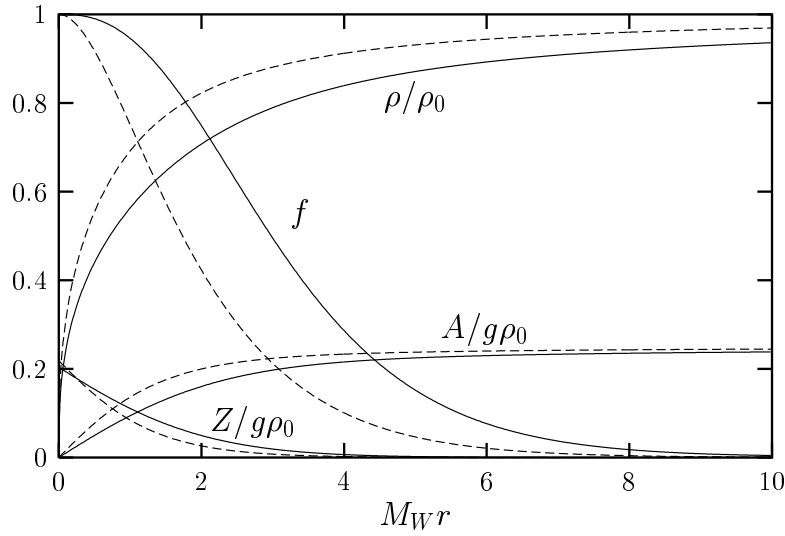


Figure 3: The electroweak dyon solution. The solid line represents the finite energy dyon and dotted line represents the Cho-Maison dyon, where we have chosen $\lambda/g^2 = 0$ and $A_0 = M_W/2$.

$$\begin{aligned}
& + \frac{1}{2}(\partial_i \rho)^2 + (1 + \alpha)^2 e^2 \rho^2 |W_i|^2 \Big\} \\
= & \int d^3 x \Big\{ \left| \epsilon_{ijk} D_j W_k \pm i(1 + \alpha) e \rho W_i \right|^2 \\
& + \frac{1}{2} \left(\partial_i \rho \mp \epsilon_{ijk} \left[i(1 + \alpha) e W_j^* W_k - \frac{g}{2e} F_{jk} \right] \right)^2 \Big\} \\
& \mp \frac{1}{e} \int d^3 x \partial_i \left[\epsilon_{ijk} \left(\frac{g}{2} F_{jk} - i(1 + \alpha) e^2 W_j^* W_k \right) \rho \right] \\
& \pm \frac{1}{e} \int d^3 x \frac{g}{2} \epsilon_{ijk} (\partial_i F_{jk}) \rho,
\end{aligned} \tag{3.13}$$

where we have used of the fact that $g^2(F_{ij})^2 = g'^2(G_{ij})^2$ which follows from the ansatz (3.2). The last integral gives a delta-function at the origin where $\rho = 0$ so that integral does not contribute. And the second integral can be converted to a surface integral at the spatial infinity where the second part $ig^2\epsilon_{ijk}\rho W_j^* W_k$ goes faster than $O(1/r^2)$, so that only the first part contributes. Thus the energy of the monopole is obviously bounded from below by

$$E \geq \left| \frac{1}{e} \int d^3 x \partial_i \left(\frac{g}{2} \epsilon_{ijk} F_{jk} \rho \right) \right|. \tag{3.14}$$

Furthermore this bound is saturated by the following equation,

$$\begin{aligned}
& \epsilon_{ijk} D_j W_k \pm i(1 + \alpha) e \rho W_i = 0, \\
& \partial_i \rho \mp \epsilon_{ijk} \left[i(1 + \alpha) e W_j^* W_k - \frac{g}{2e} F_{jk} \right] = 0,
\end{aligned} \tag{3.15}$$

which is very similar to the well-known Bogomol'nyi-Prasad-Sommerfield monopole equation of the Georgi-Glashow model. The first order differential equation is much more tractable than the second order field equation, and also solves it automatically.

Inserting the ansatz (3.2) into (3.15) we obtain the following Bogomol'nyi-type equation

$$\begin{aligned}
& \dot{f} \pm e(1 + \alpha) \rho f = 0, \\
& \dot{\rho} \mp \frac{1}{er^2} \left(1 - (1 + \alpha) \sin^2 \theta_w f^2 \right) = 0.
\end{aligned} \tag{3.16}$$

Let us consider the upper sign in more detail. Near the origin, we have

$$\begin{aligned}
f(r) & \simeq \frac{1}{\sqrt{(1 + \alpha) \sin^2 \theta_w}} - ar^{l+1}, \\
\rho(r) & \simeq \frac{2\sqrt{(1 + \alpha) \sin^2 \theta_w}}{el} ar^l,
\end{aligned} \tag{3.17}$$

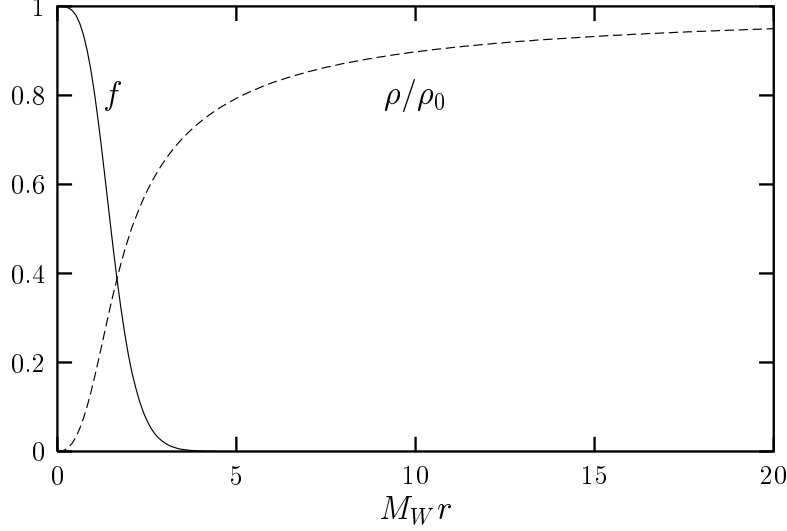


Figure 4: The analytic monopole solution in the Bogomol'nyi-Prasad-Sommerfield limit.

where $l = (-1 + \sqrt{9 + 8\alpha})/2$. On the other hand for large r , $f(r)$ approaches to zero exponentially. Thus the cloud of charged vector fields exists only in the core of monopole. Also the Higgs field has a exponentially decaying piece, with a long-range $1/r$ tail. So this solution is again very much like the Bogomol'nyi-Prasad-Sommerfield solution of the Georgi-Glashow model. In fact we can compare this with the following Bogomol'nyi-Prasad-Sommerfield equation of the Georgi-Glashow model

$$\begin{aligned} \dot{f} \pm e\rho f &= 0, \\ \dot{\rho} \mp \frac{1}{er^2}(1 - f^2) &= 0, \end{aligned} \tag{3.18}$$

which was obtained with the spherically symmetric ansatz

$$\begin{aligned} \mathbf{A}_\mu &= \frac{1}{e}(f(r) - 1)\hat{r} \times \partial_\mu \hat{r}, \\ \Phi &= \rho(r)\hat{r}, \end{aligned} \tag{3.19}$$

where Φ is the Higgs triplet of the Georgi-Glashow model. This equation has the exact solution

$$f = \frac{e\rho_0 r}{\sinh(e\rho_0 r)},$$

$$\rho = \rho_0 \coth(e\rho_0 r) - \frac{1}{er}, \quad (3.20)$$

but in our case it is not possible to express the solution in terms of the elementary functions. In Fig.4 we have plotted the Bogomol'nyi solution of (3.16) with the experimental value of $\sin^2 \theta_w$ and $f(0) = 1$.

Notice that the energy of our solution has exactly the same form as the Bogomol'nyi-Prasad-Sommerfield monopole, and is given by

$$E = \frac{4\pi}{e} \rho(\infty) = \frac{4\pi}{e^2} \sin^2 \theta_w M_W. \quad (3.21)$$

Obviously the solution is stable energetically since it is the lowest energy configuration.

B. Embedding $SU(2) \times U(1)$ to $SU(2) \times SU(2)$

As we have noticed the origin of the infinite energy of the Cho-Maison solutions was the magnetic singularity of $U(1)_{\text{em}}$. On the other hand the ansatz (2.3) also suggests that this singularity really originates from the magnetic part of the hypercharge $U(1)$ field B_μ . So one could try to obtain a finite energy monopole solution by regularizing this hypercharge $U(1)$ singularity. This could be done by introducing a hypercharged vector field to the theory. A simplest way to do this is, of course, to enlarge the hypercharge $U(1)$ and embed it to another $SU(2)$.

To construct the desired solutions we generalize the Lagrangian (3.1) by adding the following Lagrangian

$$\begin{aligned} \mathcal{L}' = & -\frac{1}{2} |\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu|^2 + ig' G_{\mu\nu} X_\mu^* X_\nu + \frac{1}{4} g'^2 (X_\mu^* X_\nu - X_\nu^* X_\mu)^2 \\ & -\frac{1}{2} (\partial_\mu \sigma)^2 - g'^2 \sigma^2 X_\mu^* X_\mu - \frac{\kappa}{4} \left(\sigma^2 - \frac{m^2}{\kappa} \right)^2, \end{aligned} \quad (3.22)$$

where X_μ is a hypercharged vector field, σ is a Higgs field, and $\tilde{D}_\mu X_\nu = (\partial_\mu + ig' B_\mu) X_\nu$. Notice that, if we introduce a hypercharge $SU(2)$ gauge field \mathbf{B}_μ and a scalar triplet Φ and identify

$$X_\mu = \frac{1}{\sqrt{2}} (B_\mu^1 + iB_\mu^2),$$

$$\begin{aligned}
B_\mu &= B_\mu^3, \\
\Phi &= (0, 0, \sigma),
\end{aligned} \tag{3.23}$$

the above Lagrangian becomes identical to

$$\mathcal{L}' = -\frac{1}{2}(\tilde{D}_\mu \Phi)^2 - \frac{\kappa}{4}\left(\Phi^2 - \frac{m^2}{\kappa}\right)^2 - \frac{1}{4}G_{\mu\nu}^2, \tag{3.24}$$

in the unitary gauge. This clearly shows that Lagrangian (3.22) is nothing but the embedding of the hypercharge $U(1)$ to an $SU(2)$ Georgi-Glashow model.

From (3.1) and (3.22) one has the following equations of motion

$$\begin{aligned}
\partial_\mu(\partial_\mu \rho) &= \frac{1}{2}g^2 W_\mu^* W_\mu \rho + \frac{1}{4}(g' B_\mu - g A_\mu)^2 \rho + \lambda\left(\frac{\rho^2}{2} + \frac{\mu^2}{\lambda}\right)\rho, \\
D_\mu(D_\mu W_\nu - D_\nu W_\mu) &= ig F_{\mu\nu} W_\mu - g^2 W_\mu(W_\nu W_\mu^* - W_\nu^* W_\mu) + \frac{1}{4}g^2 \rho^2 W_\nu, \\
\partial_\mu F_{\mu\nu} &= \frac{1}{4}g\rho^2(g A_\nu - g' B_\nu) + ig\left(W_\mu^*(D_\mu W_\nu - D_\nu W_\mu) - (D_\mu W_\nu^* - D_\nu W_\mu^*)W_\mu\right) \\
&\quad + ig\partial_\mu(W_\mu^* W_\nu - W_\nu^* W_\mu), \\
\partial_\mu G_{\mu\nu} &= \frac{1}{4}g'\rho^2(g' B_\nu - g A_\nu) + ig'\left(X_\mu^*(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu) - (\tilde{D}_\mu X_\nu^* - \tilde{D}_\nu X_\mu^*)X_\mu\right) \\
&\quad + ig'\partial_\mu(X_\mu^* X_\nu - X_\nu^* X_\mu), \\
\partial_\mu(\partial_\mu \sigma) &= 2g'^2 X_\mu^* X_\mu \sigma + \kappa\left(\sigma^2 - \frac{m^2}{\kappa}\right)\sigma, \\
\tilde{D}_\mu(\tilde{D}_\mu X_\nu - \tilde{D}_\nu X_\mu) &= ig' G_{\mu\nu} X_\mu - g'^2 X_\mu(X_\mu^* X_\nu - X_\nu^* X_\mu) + (g')^2 \sigma^2 X_\nu
\end{aligned} \tag{3.25}$$

Now for a static spherically symmetric ansatz we choose (3.2) and assume

$$\begin{aligned}
\sigma &= \sigma(r), \\
X_\mu &= \frac{i}{g'} \frac{h(r)}{\sqrt{2}} e^{i\varphi} (\partial_\mu \theta + i \sin \theta \partial_\mu \varphi).
\end{aligned} \tag{3.26}$$

With the spherically symmetric ansatz (3.25) is reduced to

$$\ddot{f} - \frac{f^2 - 1}{r^2} f = \left(\frac{g^2}{4} \rho^2 - A^2\right) f,$$

$$\ddot{\rho} + \frac{2}{r}\dot{\rho} - \frac{f^2}{2r^2}\rho = -\frac{1}{4}(B-A)^2\rho + \lambda\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)\rho,$$

$$\ddot{A} + \frac{2}{r}\dot{A} - \frac{2f^2}{r^2}A = \frac{g^2}{4}\rho^2(A-B), \quad (3.27)$$

$$\ddot{h} - \frac{h^2-1}{r^2}h = (g'^2\sigma^2 - B^2)h, \quad (3.28)$$

$$\ddot{\sigma} + \frac{2}{r}\dot{\sigma} - \frac{2h^2}{r^2}\sigma = \kappa\left(\sigma^2 - \frac{m^2}{\kappa}\right)\sigma,$$

$$\ddot{B} + \frac{2}{r}\dot{B} - \frac{2h^2}{r^2}B = \frac{g'^2}{4}\rho^2(B-A).$$

Notice that the energy of the above configuration is given by

$$E = E_W + E_X, \quad (3.29)$$

$$\begin{aligned} E_W &= \frac{4\pi}{g^2} \int_0^\infty dr \left\{ (\dot{f})^2 + \frac{(1-f^2)^2}{2r^2} + \frac{1}{2}(r\dot{A})^2 + f^2 A^2 \right. \\ &\quad \left. + \frac{g^2}{2}(r\dot{\rho})^2 + \frac{g^2}{4}f^2\rho^2 + \frac{g^2 r^2}{8}(B-A)^2\rho^2 + \frac{\lambda g^2 r^2}{2}\left(\frac{\rho^2}{2} - \frac{\mu^2}{\lambda}\right)^2 \right\} \\ &= \frac{4\pi}{g^2} C_1(\lambda/g^2) M_W, \end{aligned}$$

$$\begin{aligned} E_X &= \frac{4\pi}{g'^2} \int_0^\infty dr \left\{ (\dot{h})^2 + \frac{(1-h^2)^2}{2r^2} + \frac{1}{2}(r\dot{B})^2 + h^2 B^2 \right. \\ &\quad \left. + \frac{g'^2}{2}(r\dot{\sigma})^2 + g'^2 h^2 \sigma^2 + \frac{\kappa g'^2 r^2}{4}(\sigma^2 - \sigma_0^2)^2 \right\} \\ &= \frac{4\pi}{g'^2} C_2(\kappa/g'^2) M_X, \end{aligned}$$

where $M_W = g\rho_0/2$, and $M_X = g'\sigma_0 = g'\sqrt{m^2/\kappa}$. The boundary conditions for a regular field configuration can be chosen as

$$f(0) = h(0) = 1, \quad A(0) = B(0) = \rho(0) = \sigma(0) = 0,$$

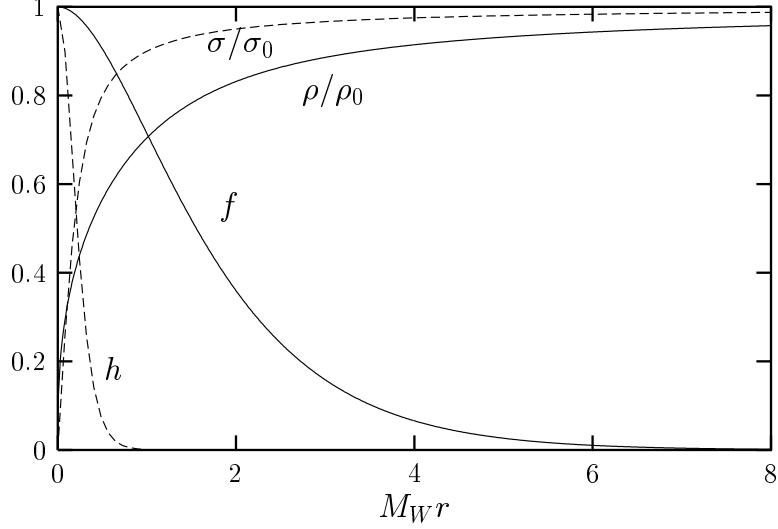


Figure 5: The $SU(2) \times SU(2)$ monopole solution, where the dashed line represents hypercharge part which describes Bogomol'nyi-Prasad-Sommerfield solution.

$$f(\infty) = h(\infty) = 0, \quad A(\infty) = B(\infty) = A_0, \quad \rho(\infty) = \rho_0, \quad \sigma(\infty) = \sigma_0. \quad (3.30)$$

Notice that the origin of the condition $B(0) = 0$ is the same as $A(0) = 0$. With the boundary condition (3.30) one may try to find the desired solution. From the physical point of view one could assume $M_X \gg M_W$, where M_X is an intermediate scale which lies somewhere between the grand unification scale and the electroweak scale. Now, let $A = B = 0$ for simplicity. Then (3.29) decouples to describes two independent systems so that the monopole solution has two cores, the one with the size $O(1/M_W)$ and the other with the size $O(1/M_X)$. With $M_X = 10M_W$ we obtain the solution shown in Fig.5 in the limit $\lambda = \kappa = 0$. In this limit we find $C_1 = 1.946$ and $C_2 = 1$ so that the energy of the solution is given by

$$E = \frac{4\pi}{e^2} \left(\cos^2 \theta_w + 0.195 \sin^2 \theta_w \right) M_X. \quad (3.31)$$

Clearly the solution describes the Cho-Maison monopole whose singularity is regularized by a Prasad-Sommerfield monopole of the size $O(1/M_X)$.

It must be emphasized that even if the energy of the monopole is fixed by the intermediate scale, the monopole should be interpreted as an electroweak monopole. To see this notice

that the size of the monopole is fixed by the electroweak scale. Furthermore from the outside the monopole looks exactly the same as the Cho-Maison monopole. Only the inner core is regularized by the hypercharged vector field. This justifies it as an electroweak monopole.

IV. Conclusions

In this paper we have discussed two ways to regularize the Cho-Maison monopole and dyon solutions of the Weinberg-Salam model, and explicitly constructed genuine finite energy electroweak monopole and dyon solutions which are analytic everywhere including the origin. The finite energy solutions are obtained with a simple modification of the interaction of the W boson or with the embedding of the hypercharge $U(1)$ to a compact $SU(2)$. It has generally been believed that the finite energy monopole must exist only at the grand unification scale [10]. But our result tells that this belief is unfounded, and endorses the existence of a totally new class of electroweak monopole whose mass is much smaller than the monopoles of the grand unification. Obviously the electroweak monopoles are topological solitons which must be stable.

Strictly speaking the finite energy solutions are not the solutions of the Weinberg-Salam model, because their existence requires a generalization of the model. But from the physical point of view there is no doubt that they should be interpreted as the electroweak monopole and dyon, because they are really nothing but the regularized Cho-Maison solutions whose size is fixed at the electroweak scale. In spite of the fact that the Cho-Maison solutions are obviously the solutions of the Weinberg-Salam model one could try to object them as the electroweak dyons under the presumption that the Cho-Maison solutions could be regularized only at the grand unification scale. Our work shows that this objection is groundless, and assures that it is not necessary for us to go to the grand unification scale to make the energy of the Cho-Maison solutions finite. This really reinforces the Cho-Maison dyons as the electroweak dyons which must be taken seriously.

Another important aspect of our result is that, unlike the Dirac monopole, the magnetic charge of the electroweak monopoles must satisfy the Schwinger quantization condition $q_m =$

$4\pi n/e$. The electroweak unification simply forbids the electromagnetic monopole with $q_m = 2\pi/e$. This is because the $U(1)_{\text{em}}$ is defined with the $U(1)$ subgroup of $SU(2)$, which affects the magnetic charge. So within the framework of the electroweak unification the unit of the magnetic charge must be $4\pi/e$.

We close with the following remarks:

- 1) The electromagnetic regularization of the Dirac monopole with the charged vector fields is nothing new. In fact it was this regularization which made the energy of the 't Hooft-Polyakov monopole finite. Furthermore it has been known that the 't Hooft-Polyakov monopole is the only analytic solution (with $\alpha = \beta = 0$) which one could obtain with this technique [11]. What we have shown in this paper is that the same technique also works to regularize the Cho-Maison solutions, but with nonvanishing α and β .
- 2) The introduction of the additional interactions (3.3) and (3.12) to the Lagrangian (2.1) could spoil the renormalizability of the Weinberg-Salam model (although this issue has to be examined in more detail). How serious would this offense, however, is not clear at this moment. Here we simply notice that the introduction of a non-renormalizable interaction (like a gravitational interaction) has been an acceptable practice to study finite energy classical solutions.
- 3) The embedding of the electroweak $SU(2) \times U(1)$ to a larger $SU(2) \times SU(2)$ or $SU(2) \times SU(2) \times U(1)$ could naturally arise in the left-right symmetric grand unification models, in particular in the $SO(10)$ grand unification, although the embedding of the hypercharge $U(1)$ to a compact $SU(2)$ may turn out to be too simple to be realistic. Independent of the details, however, our discussion suggests that the electroweak monopoles at an intermediate scale M_X could be possible in a realistic grand unification.

Certainly the existence of the finite energy electroweak monopoles should have important physical implications [12]. We will discuss on the physical implications of the electroweak monopoles separately.

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